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CONTROL DESIGN AND MINIMAL REALIZATION OF DYNAMICAL SYSTEMS

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Abstract. In the context of control synthesis the determination of the model of the plant the reduced order model and the design of the control law are not independent problems. In this paper, the minimal realization, reduced order models, in state-space, will be obtained by matching a subset of the frequency and power moments. High and low frequency moments are being simultaneously matched. Matching low frequency moments guarantee that the steady-state values will be preserved. For instance, the steady-state value to a step response is matched when the first low frequency moment is matched.

1. INTRODUCTION

A measure of the complexity of the system model is the number of first order equations used to describe it. This number is often referred as the *order* of the model. Models with elevated order are able to describe very complex phenomena. Consequently, models with high order may be required in order to provide an accurate description of a dynamic system. If the capacity of a model to accurately describe a system seems to increase with the order of the model, in practice, models with low orders are required in many situations. In some cases, the amount of information contained in a complex model may obfuscate simple, insightful behaviors, which can be better captured and explored by a model with low order. In cases such as control design and filtering, where the design procedures might be computationally very demanding, limited computational resources certainly benefit from low order models [1], [3].

Given a linear system the fastest and simplest way to produce a reduced order model is by truncation. For instance, a “natural” frequency domain model reduction procedure is to truncate poles and zeros. While deleting poles and zeros may be simple in SISO (Single-Input-Single-Output) systems, the many subtleties involving the definition of zeros imposes some extra difficulties for MIMO (Multiple-Input-Multiple-Output) systems. In this aspect, state-space methods seem to provide a more adequate framework.

2. CONTROLLABILITY AND MIMIMAL REALIZATION

In the context of model reduction, and given a state-space realization (A, B, C, D) of order n, one might wonder whether the given realization is minimal in the sense that there exists no other transfer equivalent realization (A_r, B_r, C_r, D_r) of order n_r with n_r smaller than n. The state-space realization (A, B, C, D) is minimal if, and only if, it is controllable and observable. In this case, all transfer equivalent realizations are related by a similarity transformation. The order of a minimal realization is called the minimal degree. An immediate implication is that if a given realization is not minimal, one should be able to obtain a transfer equivalent realization with reduced order. A transfer function with minimal degree is obtained when (A_r, B_r, C_r, D_r) is controllable and observable, [3]. A constructive procedure to compute such a minimal realization is based on the calculation of the controllable and observable subspaces [5], [4].

Consider a linear, continuous-time invariant (LTI) dynamical systems of order $n < \infty$ given in a minimal state-space model (*A, B, C, D*)

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t),\ x(0) = x_0 \end{aligned} \tag{1}$$

where $A \in \mathrm{R}^{n\times n}$ is the state matrix, $B \in \mathrm{R}^{n\times m}$, $C \in \mathrm{R}^{p\times m}$, and $x_0 \in \mathrm{R}^{n}$ is initial state of the system. Here, *n* is the order (or the state space dimension) of the system. The associate transfer function matrix (TFM) obtained from taking Laplace transforms in (1) and assuming $x_0 = 0$ is

$$G(s) = C(sI_n - A)^{-1}B + D \tag{2}$$

In model reduction we are faced with the problem of finding a reduced-order LTI system

$$\begin{aligned} \dot{x}_r(t) &= A_r x_r(t) + B_r u(t) \\ y_r(t) &= C_r x_r(t) + D_r u(t),\ x_r(0) = x_{r0} \end{aligned} \tag{3}$$

of order $r, r << n$, and associated TMF $G_r(s) = C_r(sI_n - A_r)^{-1}B_r + D_r$ which approximate $G(s)$ such that the predicted input-output behavior is close, e.g., $\|G - G_r\|_\infty$ or $\|G - G_r\|_2$ are small.

3. FREQUENCY MOMENTS

Assume that the transfer function *G*(*s*) is strictly proper (*D* = 0) and analytic on the imaginary axis. Its Fourier power series expansion around $s = j\omega,\ 0 \le \omega < \infty$ provides

$$G(s) = C(sI - A)^{-1}B = \sum_{i=0}^{\infty} M_i(j\omega)(s - j\omega)^i \tag{4}$$

where the matrices

$$M_i(j\omega) := C(j\omega I - A)^{-(i+1)}B,\ i = 0,1,... \tag{5}$$

are known as the low frequency moments of the transfer function *G*(s). The high frequency moments

$$M_i(j\infty) := \lim_{\omega\to\infty} M(j\omega) = CA^iB,\ i = 0,1,... \tag{6}$$

can be obtained by performing a Laurent expansion around s = 0. The high frequency moments $M_i(j\omega), i = 0,1,2...$ are also called Markov parameters, [2].
Notice that the frequency moments are input-output properties and should remain invariant under a similarity transformation.

4. MATCHING FREQUENCY AND POWER MOMENTS

In Sections 3 shown that the high and low frequency and power moments are input-output properties that remain invariant under a similarity transformation. Therefore, the minimal transfer equivalent realization which preserves the transfer functions, also preserves these parameters. In this section, reduced order models that do not preserve transfer function equivalence will be obtained by matching a subset of the frequency and power moments. Matching low frequency moments guarantee that the steady-state values will be preserved. For instance, the steady-state value to a step response is matched when the first low frequency moment is matched (see example). High and low frequency moments can be simultaneously matched as in the next algorithm:

Algorithm 1. Given the minimal and asymptotically stable-space realization (*A, B, C,* 0) of order *n,* follow the steps:

Step1. Calculate the singular value decomposition

$$\begin{bmatrix} W_q \\ W_p \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma V_1^T \tag{7}$$

where $U_1 U_1^T + U_2 U_2^T = I,\ V_1 V_1^T + V_2 V_2^T = I,$ and $\Sigma = \mathrm{diag}(\sigma_1,....\sigma_{n_r}) > 0$, and matrices W_p and W_q are defined, respectively by:

$$W_q := \begin{bmatrix} C & CA & . & . & CA^{q-1} \end{bmatrix}^T \qquad W_p := \begin{bmatrix} CA^{-1} & CA^{-2} & . & . & CA^{-p} \end{bmatrix}^T \quad .(8)$$

Step2. The matrices

$T_1 := \mathcal{G}_C V_1 (V_1^T \mathcal{G}_C V_1)^{-1}$, $R_1 := V_1$ produce the state-space realization (4) with

$$(A_r, B_r, C_r, D_r) = (A_{11}, B_1, C_1, D) = (R_1^T A T_1, R_1^T B, C T_1, D) \tag{9}$$

of order n_r. If the reduced order system has a nonsingular matrix A_r, then it is asymptotically stable and matches the first *p* low frequency moments $M_i(j0)$and power moments $R_i(j0)$, $i = 0,....,p-1$ and the first high frequency moments $M_i(j\infty)$ and power moments $R_i(j\infty)$, $i = 0,....,q-1$ of (*A, B, C,* 0).

These results can be extended to cope with moments evaluated at any set of finite frequencies. It is interesting to notice that the projections generate reduced order models that are not guaranteed to approximate the original system according to any system norm. On the other hand, it gives the engineer the opportunity to tune up the reduced order model to an specific application by arbitrarily selecting the appropriate sets of frequencies and the order of moments that are significant to the problem. Furthermore, as the examples in Section 4.2 illustrate, this does not prevent the model reduction error to have a small norm.

4.1. Control Problems and Component Cost Analysis

Several system and control problems are formulated with the objective of monitoring or controlling the system output covariance. The trace of the output covariance is referred in this section simply as the cost function. When the system disturbance inputs are modeled as impulses (white noise), these problems can be seen as equivalent to the minimization of the H_2 norm of the transfer function from the disturbances to the outputs. This justifies the need to develop reduced order models that are able to preserve the H_2 norm of the original system. The analysis of this model reduction problem is significantly simplified when the original system is transformed into the particular set of coordinates defined below.

The asymptotically stable and time-invariant state-space realization (*A,B,C,0*) of order *n*, is said to be in *cost decoupled coordinates* if the H_2 norm of the transfer function *G*(s) (cost function) can be expressed in the form

$$\|G(s)\|_2^2 = \sum_{i=1}^{n} \alpha_i \ , \ \alpha_i > 0, i = 1,.., n \tag{10}$$

The scalars $\alpha_i, i = 1,.., n$ represent independent contributions of each state to the cost function. For a given linear system, there exists more than one realization whose coordinates qualify as cost decoupled coordinates. The following lemma provides two of these realizations.

Lemma 1. An asymptotically stable and time-invariant realization (*A, B, C, 0*) is in cost decoupled coordinates if

a) the controllability Grammian $\mathcal{G}_C$ given by solving Lyapunov equation $A\mathcal{G}_C + \mathcal{G}_C A^T + BB^T = 0$ is diagonal, or

b) the matrix $\mathcal{G}_C C^T C$ is diagonal

is in cost decoupled coordinates.

Proof: The proof of this lemma consists in finding the independent factors $\alpha_i, i = 1,.., n$ mentioned in (18). In a), matrix $\mathcal{G}_C$ is diagonal, therefore

$\|G(s)\|_2^2 = trace(C\mathcal{G}_C C^T) = \sum_{i=1}^{n} (\mathcal{G}_C)_{ii} c_i c_i^T = \sum_{i=1}^{n} \alpha_i$, where $\alpha_i = (\mathcal{G}_C)_{ii} c_i c_i^T$ and the row vector c_i denotes the *i*th row of matrix *C*. In *b*), matrix $\mathcal{G}_C C^T C$ is diagonal and the cost function can be calculated as

$$\|G(s)\|_2^2 = trace(\mathcal{G}_C C^T C) = \sum_{i=1}^{n} (\mathcal{G}_C C^T C)_{ii} = \sum_{i=1}^{n} \alpha_i \ , \quad \alpha_i = (\mathcal{G}_C C^T C)_{ii} \tag{11}$$

The next lemma shows how to transform a given realization into a realization with cost decoupled coordinates that simultaneously satisfies items a) and b) of Lemma 1.

Lemma 2. Given the controllable and asymptotically stable realization (*A,B,C,0*) of order *n*, compute the symmetric and positive definite controllability Grammian $\mathcal{G}_C$ Calculate a nonsingular matrix $F \in R^{n \times n}$ such that $\mathcal{G}_C = F^T F$ and singular value decomposition $FC^T CF^T = U \Lambda U^T$ where $UU^T = I$ $\Lambda = diag(\alpha_1,..., \alpha_n) \geq 0$. The similarity transformation $x = Tz, z = T^{-1}x$ where $T = F^T U$ puts the original realization in cost decupled coordinate.

For model reduction, the previous lemma can be used to transform the original system into cost decoupled coordinates while truncation follows as in the following algorithm.

Algorithm2 Given the asymptotically stable-and controllable time-invariant (LTI) space-state realization (*A, B, C,* 0) of order *n,* follow the steps:

Step1. Calculate the controlabillity Grammian $\mathcal{G}_C$ and compute matrix $F \in R^{n \times n}$ as in $\mathcal{G}_C = F^T F$.

Step 2. Compute the singular value decomposition

$$F^T CC^T F = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} \tag{12}$$

where

$U_1 U_1^T + U_2 U_2^T = I$, $\Lambda_1 = \text{diag}(\alpha_1, \ldots \alpha_{n_r}) \geq 0$ and $\Lambda_2 = \text{diag}(\alpha_{n_r+1}, \ldots \alpha_n) \geq 0$.

Step 3. The matrices $T_1 = F^T U_1$, $T_2 = F^{-1} U_1$ produce order state realization (3) with (9) of order n_r.

4.2. Illustrative Example

The use of the model reduction procedures discussed so far will be illustrated by one example, where the dynamical controlled system is the third order stable SISO system
A=[-2 3 0;-1 -1 1;2 -3 -4]; B=[-2;2;4]; C=[1 0 0]; D=0
with transfer function:

$$G(s) = \frac{-2(s-2.46)(s+4.46)}{(s+2)(s^2+5s+10)} \tag{13}$$

Table 1 H_2 and H_∞ norms of the model error

Model	$G_r(s)$	$\|G_r(s)\|_2$	$\|G_r(s)\|_\infty$	$\|G(s)-G_r(s)\|_2$	$\|G(s)-G_r(s)\|_\infty$
full	See (.)	1.21	1.09	-	-
# 1, # 2	$\frac{3.143}{s+2.857}$	1.21	1.09	1.85	1.29
# 3	$\frac{2.31}{s+1.65}$	1.27	1.39	1.57	1.10

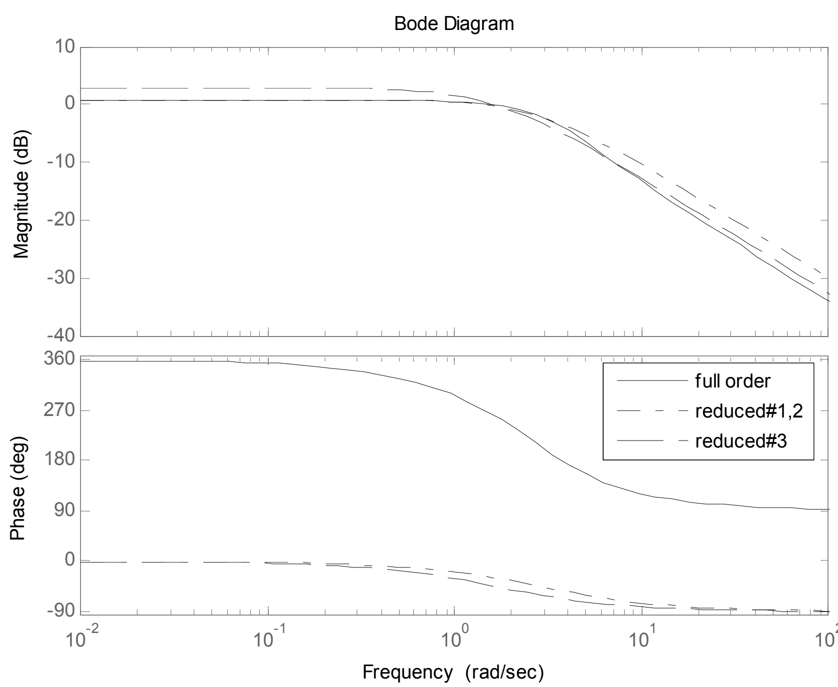
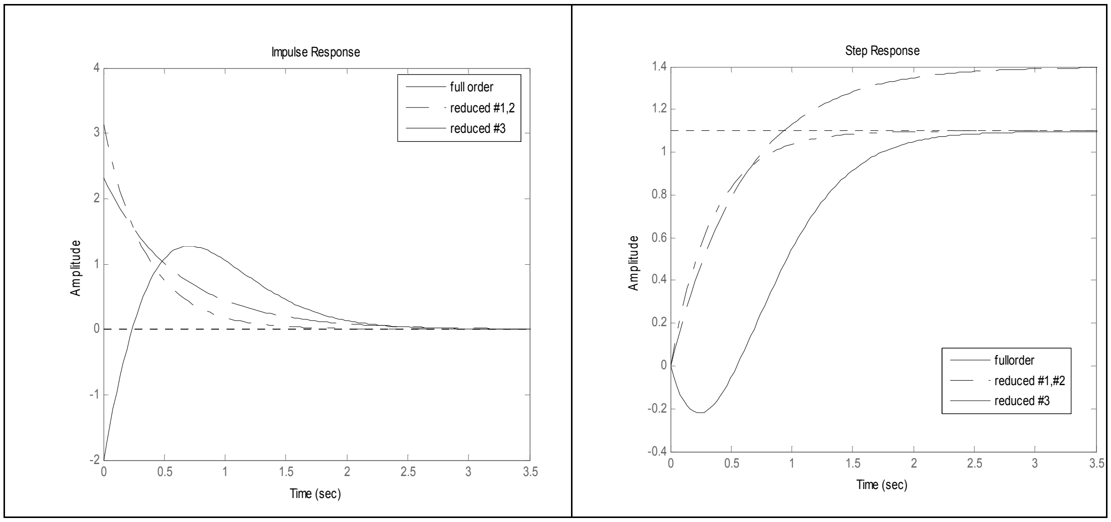


Figure 1 Bode plot: magnitude and phase



(a) ***(b)***

Figure 2 Reduce order models of order one

Notice that this transfer function has a zero on the right half of the complex plane, hence it is non-minimum phase. First, three different reduced order models of order one have been produced using the techniques introduced in this paper. The bode diagrams of the full order model and the reduced order models is depicted in Figures 1. The impulse response and the step response are given respectively in Figure 2 (a) and 2 (b). All the obtained models and the H_2 and $H\infty$ norms of the model error are given in Table 1.
Some comments on the generation of the reduced order models and their performance follow.

Reduced order model #1 (component cost analysis) has been obtained using Algorithm 1 with $n_r = 1$. Notice that CC^T has rank one and, as expected, the reduced order model matches exactly the H_2. Notice that this does not guarantee any property on the H_2 norm of the model reduction error. Indeed, Table 1 reveals that this reduced order model is outperformed by all other models.

The reduced order model #2 has been obtained using Algorithm 2 with q = 1, p = 0. That is, only the first high-frequency moment (Markov parameter) of system (13) has been matched. As the system is non minimum phase, the obtained reduced order model has phase 180deg at $\omega = 0$. In this example, the reduced model #1, designed by component cost analysis, is coincidently the same as model #2. Notice that in the design of the reduced order model #1 only the first high-frequency power moment (and not the first Markov parameter) is guaranteed to be matched and the two distinct methods should provide different models in more complex situations.

The reduced order model #3 has been obtained using Algorithm 2 with q = 0, p = 1. The first low frequency moment has been matched, which guarantees that the steady-state value of the step response.

5. CONCLUSIONS

It is important to stress that the use of the results require the careful intervention of the engineer in the selection of the system output. In the most common situations, an engineering system will have less outputs than its order. Hence, the square matrix $\mathcal{G}_C C^T C$ will be, typically, rank deficient. For single-output systems, for example, it will have rank one and, regardless of the order of the original system, Algorithm 2 is able to produce a reduced order model of order one that perfectly matches the H_2 norm of the original system $\alpha_i = 0, i = 2,..,n$. This fact reinforces the need for a good selection of the system output, which should be able to capture the important aspects of the system behavior. An appropriate output for model reduction will often be constituted by the measured output of the original system augmented by some of its derivatives and other selected important signals which are internal to the model.

Matching the first *q* Markov parameters guarantee that the first *q* time moments of the impulse response are matched. The preservation of this feature is especially important in non-minimum phase systems. For instance, the response of a non-minimum phase system to a positive step might present at time *t* = 0 a negative derivative. This behavior can be captured by matching high frequency moments.

Matching low frequency moments guarantee that the steady-state values will be preserved. For instance, the steady-state value to a step response is matched when the first low frequency moment is matched. High and low frequency moments can be simultaneously matched.

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